JOURNAL OF APPROXIMATION THEORY 64, 55–68 (1991)

Tietze Extensions and Continuous Selections for Metric Projections

FRANK DEUTSCH, WU LI, AND SUNG-HO PARK*

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802

Communicated by Günther Nürnberger

Received June 30, 1989

There is an intimate relationship between (1) the set of all Tietze extensions of a given continuous function on a compact subset S of a locally compact Hausdorff space T to all of T, and (2) the set of all best approximations to elements of $C_0(T)$ from the ideal M in $C_0(T)$ consisting of those functions which vanish on S. This relation is used, for example, to deduce that the Tietze extension map has a linear selection if and only if the metric projection onto M has a linear selection. It is known that the former holds whenever T is metrizable.

1. INTRODUCTION

Let T be a locally compact Hausdorff space and S a compact subset of T. The Tietze extension theorem (cf. [11, Theorem 20.4]) states that each real continuous function g on S has a continuous extension $\tilde{g}$ to all of T which vanishes off a compact set and has the same norm: $\max\{|g(s)| \mid s\in S\} = \max\{|\tilde{g}(t)| \mid t\in T\}$. In particular, the extension is in $C_0(T)$, the space of all real continuous functions f "vanishing at infinity" (i.e., $\{t\in T \mid |f(t)| \geqslant \varepsilon\}$ is compact for each $\varepsilon > 0$), and endowed with the supremum norm

$$\|f\| = \sup\{|f(t)| \mid t\in T\}.$$

If T is actually compact, then $C_0(T)$ reduces to the space of all real continuous functions on T, and is usually denoted $C(T)$. For any $g\in C(S)$, we write

$$\|g\|_S := \sup\{|g(s)| \mid s\in S\}.$$

* Present address: Department of Mathematics, Sogang University, C.P.O. 1142, Seoul 121-110, Korea.

0021-9045/91 $3.00

1.1. DEFINITION. For each $g \in C(S)$, let $E(g)$ denote the set of all *Tietze extensions* of g to $C_0(T)$. That is,

$$E(g) = \{f \in C_0(T) \mid f|_S = g,\ \|f\| = \|g\|_S\}.$$

In this notation, the Tietze extension theorem simply states that $E(g)$ is not empty for each $g \in C(S)$.

Next we define a subspace of $C_0(T)$ by

$$M = M_S := \{f \in C_0(T) \mid f|_S = 0\}.$$

It is easy to see that M is a closed ideal in $C_0(T)$.

1.2. DEFINITION. For each $f \in C_0(T)$, the set of all *best approximations* to f from M is defined by

$$P_M(f) := \{g \in M \mid \|f - g\| = d(f, M)\},$$

where

$$d(f, M) := \inf\{\|f - g\| \mid g \in M\}.$$

It is a well-known result of Alfsen and Effros [1] that $P_M(f)$ is not empty for each $f \in C_0(T)$. (This is also an immediate consequence of Theorem 3.3 below.)

In Section 2 we prove that the set-valued mapping E is a contraction and admits a continuous homogeneous selection. The main result of Section 3 (Theorem 3.3) is a formula relating E and P_M. Namely, $P_M(f) = f - E(f|_S)$ for each $f \in C_0(T)$. From this, one can deduce that P_M has a continuous selection p which also satisfies $p(\alpha f + g) = \alpha p(f) + p(g)$ for all $f \in C_0(T)$, $g \in M$, and $\alpha \in \mathbf{R}$. Also, P_M is Lipschitz continuous with Lipschitz constant 2. The condition that E have a linear selection is equivalent to P_M having a linear selection (Theorem 3.8). Using the well-known Borsuk theorem [3], we deduce that P_M has a linear selection when T is metrizable (Corollary 3.9). Finally, some results are established which relate the condition that M be complemented with the existence of various types of selections for E and P_M. In particular, we have shown (Theorem 3.10) that if M is complemented, then P_M has a Lipschitz continuous selection.

2. TIETZE EXTENSIONS

In this section we establish a few properties of the Tietze extension map E.

2.1. LEMMA. (1) *For each $g \in C(S)$, $E(g)$ is a (nonempty) closed, bounded, and convex subset of $C_0(T)$.*

(2) *E is "homogeneous"; i.e., $E(\alpha g) = \alpha E(g)$ for each $g \in C(S)$ and $\alpha \in \mathbf{R}$.*

Proof. (1) Simple to prove.

(2) Let $g \in C(S)$ and $\alpha > 0$. Then

$$\begin{aligned} E(\alpha g) &= \{f \in C_0(T) \mid f|_S = \alpha g, \ \|f\| = \|\alpha g\|_S\} \\ &= \alpha\{\alpha^{-1}f \mid f \in C_0(T), \alpha^{-1}f|_S = g, \ \|\alpha^{-1}f\| = \|g\|_S\} \\ &= \alpha\{f \in C_0(T) \mid f|_S = g, \ \|f\| = \|g\|_S\} = \alpha E(g). \end{aligned}$$

Also,

$$\begin{aligned} E(-g) &= \{f \in C_0(T) \mid f|_S = -g, \ \|f\| = \|-g\|_S\} \\ &= -\{-f \in C_0(T) \mid -f|_S = g, \ \|-f\| = \|g\|_S\} \\ &= -\{f \in C_0(T) \mid f|_S = g, \ \|f\| = \|g\|_S\} = -E(g). \end{aligned}$$

This implies that E is homogeneous. ∎

Let Y be a subspace of $C_0(T)$ and let H denote the Hausdorff metric on the space $H(Y)$ of all nonempty subsets of Y which are closed, bounded, and convex. Thus for any A, B in $H(Y)$, we have

$$H(A, B) := \max\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\}.$$

We now show that the Tietze extension mapping $E\colon C(S) \to H(C_0(T))$ is Lipschitz continuous with Lipschitz constant 1. That is, E is a contraction.

2.2. THEOREM. *For any $g, h \in C(S)$,*

$$H(E(g), E(h)) \leqslant \|g - h\|_S. \tag{2.2.1}$$

Proof. Let $f \in E(g)$. Then $f|_S = g$ and $\|f\| = \|g\|_S$. Choose any $q \in E(h - g)$. Then $q|_S = h - g$ and $\|q\| = \|h - q\|_S$. Define the notation

$$[r]_a^b := \max\{a, \min\{r, b\}\} = \begin{cases} b & \text{if } r > b \\ r & \text{if } a \leqslant r \leqslant b \\ a & \text{if } r < a \end{cases}$$

and set

$$p(t) := [f(t) + q(t)]_{-\|h\|_S}^{\|h\|_S}, \qquad t \in T.$$

It is easy to check that $p \in C_0(T)$.

Claim: $p \in E(h)$. (2.2.2)

In fact, $\|p\| \leqslant \|h\|_S$ and, for $t \in S$,

$$p(t) = [f(t)+q(t)]_{-\|h\|_S}^{\|h\|_S} = [g(t)+h(t)-g(t)]_{-\|h\|_S}^{\|h\|_S} = [h(t)]_{-\|h\|_S}^{\|h\|_S} = h(t).$$

This proves the claim.

Claim: $\|p-f\| \leqslant \|g-h\|_S$. (2.2.3)

For any $t \in T$, we consider three cases.

(i) $f(t)+q(t) \in [-\|h\|_S, \|h\|_S]$.

Then $p(t) = f(t)+q(t)$ and

$$|p(t)-f(t)| = |q(t)| \leqslant \|q\| = \|h-g\|_S. \tag{2.2.4}$$

(ii) $f(t)+q(t) > \|h\|_S$.

Then $p(t) = \|h\|_S$. But

$$\|g-h\|_S \geqslant \|g\|_S - \|h\|_S = \|g\|_S - p(t) \geqslant f(t)-p(t) > -q(t) \geqslant -\|g-h\|_S.$$

That is,

$$|f(t)-p(t)| \leqslant \|g-h\|_S. \tag{2.2.5}$$

(iii) $f(t)+q(t) < -\|h\|_S$.

Then $p(t) = -\|h\|_S$ and

$$-\|g-h\|_S \leqslant -\|g\|_S + \|h\|_S = -\|g\|_S - p(t) \leqslant f(t)-p(t) \leqslant -q(t) \leqslant \|g-h\|_S.$$

Thus

$$|f(t)-p(t)| \leqslant \|g-h\|_S. \tag{2.2.6}$$

By (2.2.4), (2.2.5), and (2.2.6) we get

$$|f(t)-p(t)| \leqslant \|g-h\|_S$$

for all $t \in T$. This proves (2.2.3).

It follows from (2.2.3) that

$$\sup_{f \in E(g)} \inf_{p \in E(h)} \|f-p\| \leqslant \|g-h\|_S$$

for any $g, h \in C(S)$. By symmetry, we also obtain

$$\sup_{p \in E(h)} \inf_{f \in E(g)} \|f-p\| \leqslant \|g-h\|_S$$

for any $g, h \in C(S)$. Thus (2.2.1) holds. ∎

A *selection* for the set-valued mapping E is any function $e: C(S) \to C_0(T)$ such that $e(g) \in E(g)$ for each $g \in C(S)$.

By Theorem 2.2, E is Lipschitz continuous and, in particular, lower semicontinuous. By Michael's theorem [10], E admits a continuous selection e. Moreover, by a result of the authors [6, Lemma 3.1], we may also choose e to be "homogeneous"; i.e.,

$$e(\alpha f) = \alpha e(f), \qquad f \in C(S), \alpha \in \mathbf{R}.$$

In fact, if e is a continuous (resp. Lipschitz continuous) selection for E, define $\tilde{e}$ on $C(S)$ by

$$\tilde{e}(f) = \begin{cases} \frac{1}{2} \|f\|_S \left[e\left(\frac{f}{\|f\|_S}\right) - e\left(\frac{-f}{\|f\|_S}\right)\right] & \text{if } f \neq 0 \\ 0 & \text{if } f = 0. \end{cases}$$

Then it can be readily verified [6] that $\tilde{e}$ is a continuous (resp. Lipschitz continuous) selection for E which is also homogeneous. The proof of this fact uses the properties that E is homogeneous and "bounded"; i.e.,

$$\sup\{\|g\| \mid g \in E(f)\} \leqslant \|f\|_S, \qquad f \in C(S).$$

These remarks can be summarized in the following corollary.

2.3. COROLLARY. *The Tietze extension map E admits a continuous homogeneous selection.*

In Section 3, we will see that a stronger result is available under certain conditions (e.g., if T is metrizable).

3. BEST APPROXIMATION FROM CLOSED IDEALS

Recall that

$$M = \{f \in C_0(T) \mid f|_S = 0\}$$

is a closed ideal in $C_0(T)$. The set-valued mapping P_M defined on $C_0(T)$ by

$$P_M(f) = \{g \in M \mid \|f - g\| = d(f, M)\}$$

is called the *metric projection* onto M. As noted in the Introduction, $P_M(f) \neq \phi$ for each $f \in C_0(T)$ by [1]. Since it is easy to verify that $P_M(f)$ is a closed, bounded, and convex subset of M, we see that $P_M: C_0(T) \to H(M)$.

The first result is a useful distance formula from any $f \in C_0(T)$ to M.

3.1. LEMMA. *For each* $f \in C_0(T)$,

$$d(f, M) = \|f\|_S := \max\{|f(s)| \mid s \in S\}.$$

Proof. For any $g \in M$,

$$\|f - g\| \geqslant \max_{s \in S} |f(s) - g(s)| = \max_{s \in S} |f(s)|.$$

Thus

$$d(f, M) \geqslant \|f\|_S. \tag{3.1.1}$$

Since f is continuous, for any $\varepsilon > 0$ and $s \in S$ choose a neighborhood U_s of s so that

$$|f(t) - f(s)| < \varepsilon \qquad \text{for all} \quad t \in U_s.$$

Since S is compact, there exist a finite number of points $s_1, ..., s_n$ in S such that $S \subset \bigcup_1^n U_{s_i}$.

By Urysohn's lemma we can choose $\tilde{g} \in C_0(t)$ so that $\tilde{g} = 1$ on S, $\tilde{g} = 0$ off $\bigcup_1^n U_{s_i}$, and $0 \leqslant \tilde{g} \leqslant 1$. Then $g := f(1 - \tilde{g}) \in C_0(T)$, $g = 0$ on S, and so $g \in M$. If $t \in T \setminus \bigcup_1^n U_{s_i}$, $\tilde{g}(t) = 0$ so $|f(t) - g(t)| = 0$. If $t \in U_{s_i}$ for some i, then

$$|f(t) - g(t)| = |f(t)\, \tilde{g}(t)| \leqslant |f(t)| \leqslant |f(t) - f(s_i)| + |f(s_i)| < \varepsilon + \|f\|_S.$$

Hence $\|f - g\| < \|f\|_S + \varepsilon$ which implies

$$d(f, M) < \|f\|_S + \varepsilon.$$

Since ε was arbitrary,

$$d(f, M) \leqslant \|f\|_S. \tag{3.1.2}$$

Combining (3.1.1) and (3.1.2) we obtain the result. ∎

The *kernel* of P_M is the set

$$\ker P_M := \{f \in C_0(T) \mid 0 \in P_M(f)\} = \{f \in C_0(T) \mid \|f\| = d(f, M)\}.$$

An immediate consequence of Lemma 3.1 is the

3.2. COROLLARY. $\ker P_M = \{f \in C_0(T) \mid \|f\| = \|f\|_S\}$.

We now state the main result of this section. It reveals an intimate connection between the set of best approximations to f from M and the set of all Tietze extensions of $f|_S$.

3.3. THEOREM. *For each* $f \in C_0(T)$,

$$P_M(f) = f - E(f|_S).$$

In particular, best approximations from M to any $f \in C_0(T)$ always exist.

Proof. Let $g \in P_M(f)$. Then, using Lemma 3.1, $h = f - g$ satisfies

$$\|h\| = \|f - g\| = d(f, M) = \|f\|_S$$

and, for $t \in S$,

$$h(t) = f(t) - g(t) = f(t).$$

Thus $h \in E(f|_S)$ and $g \in f - E(f|_S)$.

Conversely, suppose $h \in E(f|_S)$. Setting $g = f - h$ we see that $g = 0$ on S so $g \in M$. Also,

$$\|f - g\| = \|h\| = \|f\|_S = d(f, M)$$

implies that $g \in P_M(f)$. Hence $h = f - g \in f - P_M(f)$. ∎

Remark. It is worth noting that Theorem 3.3 can also be deduced from a general existence theorem established by one of us [4, Theorem 4.2]. There it was proved that if M is any subspace of a normed linear space X, then

$$P_M(x) = x - H_{M^\perp}(x), \qquad x \in X,$$

where $H_{M^\perp}(x)$ denotes the set of all "Helly extensions" of x relative to $M^\perp$ That is,

$$H_{M^\perp}(x) = \{y \in X \mid x^*(y) = x^*(x) \text{ for every } x^* \in M^\perp, \ \|y\| = \|x\|_{M^\perp}\},$$

where $\|x\|_{M^\perp} = \sup\{x^*(x) \mid x^* \in M^\perp, \ \|x^*\| \leqslant 1\}$. If we specialize this by taking $X = C_0(T)$ and $M = \{g \in C_0(T) \mid g|_S = 0\}$, we obtain that $H_{M^\perp}(x) = E(x|_S)$ and we recover Theorem 3.3.

Let X and Y be Banach spaces and $H(Y)$ denote the collection of all nonempty subsets of Y which are closed, bounded, and convex. Endow $H(Y)$ with the Hausdorff metric H. That is, for $A, B \in H(Y)$,

$$H(A, B) := \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\},$$

where

$$d(a, B) := \inf_{b \in B} \|a - b\|.$$

A (set-valued) mapping $F: X \to H(Y)$ is called *bounded* if there is a constant c such that

$$\sup\{\|y\| \mid y \in F(x)\} \leqslant c\|x\|$$

for each $x \in X$. F is called *homogeneous* if

$$F(\alpha x) = \alpha F(x)$$

for each $x \in X$ and $\alpha \in \mathbf{R}$.

A function $f: X \to Y$ is called a *selection* for F if $f(x) \in F(x)$ for each $x \in X$. A selection is called *homogeneous* if

$$f(\alpha x) = \alpha f(x) \qquad \text{for each} \quad x \in X,\ \alpha \in \mathbf{R}.$$

If Y is a subspace of X, a selection f is called *additive modulo* Y if

$$f(x+y) = f(x) + f(y) \qquad \text{for every} \quad x \in X,\ y \in Y.$$

3.4. Corollary. *The metric projection P_M has a continuous selection which is homogeneous and additive modulo M.*

Proof. By Corollary 2.3, E has a selection e which is continuous and homogeneous. Define p on $C_0(T)$ by $p = I - e \circ R$, where I is the identity on $C_0(T)$ and $R: C_0(T) \to C(S)$ is the restriction map $Rf = f|_S$. From Theorem 3.3 it is seen that p is a selection for P_M which is continuous and homogeneous. To show that p is additive modulo M, let $f \in C_0(T)$ and $g \in M$. Then

$$\begin{aligned} p(f+g) &= f + g - e((f+g)|_S) = f + g - e(f|_S + g|_S) = f + g - e(f|_S) \\ &= p(f) + g = p(f) + p(g). \quad \blacksquare \end{aligned}$$

Fakhoury [8] and, independently, Holmes, Scranton, and Ward [9], have given nonconstructive proofs that the metric projection onto an M-ideal in a Banach space has a continuous homogeneous selection. Yost [13] has deduced, more generally, that for a certain class of subspaces M which include the M-ideals, P_M admits a continuous homogeneous selection which is additive modulo M. Since each M-ideal in $C_0(T)$ has the form

$$M = \{f \in C_0(T) \mid f|_S = 0\}$$

for some closed subset S of T, it follows that nonconstructive proofs of Corollary 3.4 were also given in [13] and (without the "additive modulo M" statement) in [8, 9].

3.5. COROLLARY. *The metric projection P_M is Lipschitz continuous*:

$$H(P_M(f), P_M(g)) \leqslant 2\|f-g\|$$

for all f, g in $C_0(T)$.

Proof. Using Theorems 3.3 and Lemma 2.2, we obtain for any f, g in $C_0(T)$,

$$\begin{aligned} H(P_M(f), P_M(g)) &= H(f-E(f|_S), g-E(g|_S)) \leqslant H(E(f|_S), E(g|_S)) + \|f-g\| \\ &\leqslant \|f-g\|_S + \|f-g\| \leqslant 2\|f-g\|. \quad ∎ \end{aligned}$$

Remarks. The constant 2 in Corollary 3.5 is best possible. This can be seen, for example, by taking $T=\{1,2\}$ and $S=\{2\}$ so $C(T)=l_\infty(2)$ is the plane and $M=\{f\in C(T) \mid f(2)=0\}$ is the "horizontal axis." Taking $f=(0,0)$ and $g=(1,1)$, we observe that $P_M(f)=0$,

$$P_M(g)=\{\rho(0,1) \mid 0\leqslant \rho \leqslant 2\},$$

and

$$H(P_M(f), P_M(g)) = 2 = 2\|f-g\|.$$

It perhaps is worth noting that Corollary 3.4 can also be deduced from Corollary 3.5, the Michael selection theorem, and Theorem 3.4 of [6].

Another consequence of Theorem 3.3 is that selections of one type for the mapping E are equivalent to selections of a similar type for P_M. Before proving this, it is convenient to isolate a key step that is used in at least three places in the sequel.

3.6. LEMMA. *Let p: $C_0(T)\to M$ be idempotent (i.e., $p^2=p$) and additive modulo M. Then*

$$f-p(f)=h-p(h) \tag{3.6.1}$$

for all $f, h\in C_0(T)$ with $f|_S=h|_S$. In particular, the function e: $C(S)\to C_0(T)$, defined by

$$e(g) := f-p(f), \qquad g\in C(S),$$

for any $f\in C_0(T)$ with $f|_S=g$, is well-defined.

Proof. Let $f, h\in C_0(T)$ and $f|_S=h|_S$. Then $g:=f-h\in M$ so

$$p(f)=p(h+g)=p(h)+p(g)=p(h)+g=p(h)+f-h.$$

This proves (3.6.1). ∎

In particular, p satisfies the hypothesis of Lemma 3.6 if p is an ordinary (i.e., linear) projection onto M, or if p is a selection for P_M which is additive modulo M.

3.7. THEOREM. *E has a linear selection (resp. Lipschitz continuous selection) if and only if P_M has a linear selection (resp. Lipschitz continuous selection which is additive modulo M).*

Proof. We prove the statement about Lipschitz continuous selections. The statement about linear selections is similar, but simpler.

Let e be a Lipschitz continuous selection for E. Then there is a constant $\lambda > 0$ such that

$$\|e(f) - e(g)\| \leqslant \lambda \|f - g\|_S \tag{3.7.1}$$

for all $f, g \in C(S)$. Define p on $C_0(T)$ by

$$p(f) := f - e(f|_S).$$

By Theorem 3.3, p is a selection for P_M. Also,

$$\begin{aligned}\|p(f) - p(h)\| &= \|f - e(f|_S) - h + e(h|_S)\| \leqslant \|f - h\| + \|e(f|_S) - e(h|_S)\| \\ &\leqslant \|f - h\| + \lambda\|f - h\|_S \leqslant (1 + \lambda)\, \|f - h\|\end{aligned}$$

implies that p is Lipschitz continuous. Further, for $f \in C_0(T)$ and $g \in M$,

$$p(f + g) = f + g - e((f + g)|_S) = f + g - e(f|_S) = p(f) + g = p(f) + p(g)$$

implies that p is additive modulo M.

Conversely, suppose p is a Lipschitz continuous selection for P_M which is additive modulo M, and having Lipschitz constant λ. Then by Lemma 3.6, the function $e\colon C(S) \to C_0(T)$ defined by

$$e(g) = f - p(f),$$

for any $f \in C_0(T)$ with $f|_S = g$, is well-defined. Moreover, by Theorem 3.3, e is a selection for E.

For $g_i \in C(S)$ $(i = 1, 2)$, choose $f_1 \in C_0(T)$ so that $f_1|_S = g_1$ and $h_1 \in E(g_2 - g_1)$. Then

$$(f_1 + h_1)|_S = g_1 + g_2 - g_1 = g_2$$

so

$$\begin{aligned}\|e(g_1) - e(g_2)\| &= \|f_1 - p(f_1) - (f_1 + h_1) + p(f_1 + h_1)\| \\ &= \|-h_1 + p(f_1 + h_1) - p(f_1)\| \\ &\leqslant \|h_1\| + \|p(f_1 + h_1) - p(f_1)\| \\ &\leqslant (1 + \lambda)\, \|h_1\| = (1 + \lambda)\, \|g_1 - g_2\|_S.\end{aligned}$$

This proves that e is Lipschitz continuous. ∎

The next result gives a useful alternate characterization of when E has a linear selection.

3.8. THEOREM. *The following statements are equivalent.*

(1) *E has a linear selection;*

(2) *P_M has a linear selection;*

(3) $\ker P_M$ *contains a closed subspace N such that $C_0(T) = M \oplus N$;*

(4) *M is complemented in $C_0(T)$, say $C_0(T) = M \oplus N$, and the projection P onto M along N satisfies $\|I - P\| = 1$.*

Proof. The equivalence of (1) and (2) is contained in Theorem 3.7, and the equivalence of (2) and (3) is from Stoer [12] and, more generally, [5, Theorem 2.2].

(3) ⇒ (4). Assume (3) holds and let P denote the projection onto M along N. Then since $I - P$ is the projection onto N, $\|I - P\| \geqslant 1$. But for all $f \in C_0(T)$, $f - P(f) \in N \subset \ker P_M$ so

$$\|(I-P)f\| = \|f - P(f)\| = d(f - P(f), M) = d(f, M) \leqslant \|f\|.$$

Hence $\|I - P\| = 1$.

(4) ⇒ (2). Suppose (4) holds. Let $f \in C_0(T)$ and choose any $g \in P_M(f)$. Then

$$\|f - P(f)\| = \|f - g - P(f - g)\| = \|(I - P)(f - g)\| \leqslant \|f - g\|$$

implies that $P(f) \in P_M(f)$. That is, P is a linear selection for P_M. ∎

It was proved by Borsuk [3] (more generally, see Dugundji [7] and Arens [2]) that *if T is* (compact and) *metrizable, then E has a linear selection.* However, their proofs are also valid in the locally compact case. This fact, along with Theorem 3.8, implies the next result.

3.9. COROLLARY. *If T is metrizable, then P_M has a linear selection and M is complemented.*

In particular, Theorem 3.8 implies that if E has a linear selection, then M is complemented. We do not know whether the converse is valid. That is, if M is complemented, must E have a linear selection? However, we do have a partial converse.

3.10. THEOREM. *If M is complemented, then E has a Lipschitz continuous homogeneous selection. In particular, P_M has a Lipschitz continuous selection which is homogeneous and additive modulo M.*

Proof. The last statement follows from Theorem 3.7 and the comment following Lemma 2.2.

Let $C_0(T) = M \oplus N$, let P denote the projection onto M along N, and let $Q = I - P$. That is, Q is the projection onto N along M. Then by Lemma 3.6, $Q(f) = Q(h)$ for each $f, h \in C_0(T)$ with $f|_S = h|_S$. Next define e on $C(S)$ by

$$e(g) = [Q(f)]^{\|g\|_S}_{-\|g\|_S}$$

for any $f \in C_0(T)$ with $f|_S = g$. [The notation $[r]^b_a$ is defined as in the proof of Lemma 2.2.] Since $f - Q(f) \in M$ for any $f \in C_0(T)$, when $f \in E(g)$ we have that

$$g = f|_S = Q(f)|_S$$

so $g = e(g)|_S$ and $e(g) \in E(g)$. That is, e is a selection for E. Next we verify that e is Lipschitz continuous.

First observe that for $a, b \geqslant 0$, it is easy to verify that

$$|[t_1]^a_{-a} - [t_2]^a_{-a}| \leqslant |t_1 - t_2| \tag{3.10.1}$$

and

$$|[t]^a_{-a} - [t]^b_{-b}| \leqslant |b - a|. \tag{3.10.2}$$

Now let $g_i \in C(S)$ $(i = 1, 2)$ and choose $f_1 \in C_0(T)$ such that $f_1|_S = g_1$, $h_1 \in E(g_2 - g_1)$, and set $f_2 = f_1 + h_1$. Then $f_2|_S = g_2$ and $\|h_1\| = \|g_2 - g_1\|_S$. Let $\lambda = \max\{\|g_1\|_S, \|g_2\|_S\}$. Then

$$0 \leqslant \lambda - \|g_i\|_S \leqslant \|g_1 - g_2\|_S \qquad (i = 1, 2). \tag{3.10.3}$$

Using (3.10.1), (3.10.2), and (3.10.3), we obtain

$$\begin{aligned}
\|e(g_1) - e(g_2)\| &= \|[Q(f_1)]^{\|g_1\|_S}_{-\|g_1\|_S} - [Q(f_2)]^{\|g_2\|_S}_{-\|g_2\|_S}\| \\
&\leqslant \|[Q(f_1)]^{\|g_1\|_S}_{-\|g_1\|_S} - [Q(f_1)]^{\lambda}_{-\lambda}\| + \|[Q(f_1)]^{\lambda}_{-\lambda} - [Q(f_2)]^{\lambda}_{-\lambda}\| \\
&\quad + \|[Q(f_2)]^{\lambda}_{-\lambda} - [Q(f_2)]^{\|g_2\|_S}_{-\|g_2\|_S}\| \\
&\leqslant |\lambda - \|g_1\|_S| + \|Q(f_1) - Q(f_2)\| - |\lambda - \|g_2\|_S| \\
&\leqslant 2\|g_1 - g_2\|_S + \|Q\| \, \|f_1 - f_2\| \\
&= 2\|g_1 - g_2\|_S + \|Q\| \, \|h_1\| = (2 + \|Q\|) \, \|g_1 - g_2\|_S.
\end{aligned}$$

This proves e is Lipschitz continuous with Lipschitz constant $2 + \|Q\|$.

Finally, by the remark following Lemma 2.2, we can arrange that e is homogeneous. ∎

There are cases in which P_M has a linear selection and M is complemented, which do not require the metrizability of T. This is when either S or $T\backslash S$ is finite. That is, when M is either finite-codimensional or finite-dimensional.

3.11. COROLLARY. *Let $S=\{s_1, ..., s_n\}$ be a finite subset of the locally compact Hausdorff space T and let*

$$M=\{f\in C_0(T)\,|\,f(s_i)=0\ (i=1, 2, ..., n)\}.$$

Then E has a linear selection given by

$$e(g):=\sum_1^n f(s_i)\,x_i, \qquad g\in C(S) \tag{3.11.1}$$

and P_M has a linear selection given by

$$p(f)=f-\sum_1^n f(s_i)\,x_i, \qquad f\in C_0(T), \tag{3.11.2}$$

where $\{x_1, x_2, ..., x_n\}$ is any prescribed set in $C_0(T)$ having the property that $0\leqslant x_i\leqslant 1$, $x_i(t_i)=1$, and the supports of x_i and x_j are disjoint if $i\neq j$.

Proof. The existence of the functions x_i is guaranteed by Urysohn's lemma. Next note that the mapping e on $C(S)$ defined by (3.11.1) satisfies $e: C(S)\to C_0(T)$, e is linear, and $e(g)\in E(g)$ for each $g\in C(S)$. By Theorem 3.3, the map p defined by (3.11.2) is a linear selection for P_M. ∎

3.12. COROLLARY. *Let S be a compact subset of the (locally) compact Hausdorff space T such that $T\backslash S$ is finite, and let*

$$M=\{f\in C_0(T)\,|\,f|_S=0\}.$$

Then E has a linear selection e given by

$$e(g)(t)=\begin{cases} g(t) & \text{if } t\in S \\ 0 & \text{if } t\in T\backslash S\end{cases}, \qquad g\in C(S) \tag{3.12.1}$$

and P_M has a linear selection p defined by

$$p(f)=f-f\chi_S, \qquad f\in C_0(T), \tag{3.12.2}$$

where χ_S is the characteristic function of S.

Proof. Since S is both open and closed, $\chi_S\in C_0(T)$ and $e(g)\in C_0(T)$ for each $g\in C(S)$. The remainder of the proof is like that of Corollary 3.11. ∎

The last two corollaries along with Corollary 3.9 raise the natural question: Must P_M (or equivalently E) *always* have a linear selection?

References

1. E. M. Alfsen and E. G. Effros, Structure in real Banach space, I, *Ann. of Math.* (2) **96** (1972), 98–128.
2. R. Arens, Extension of functions on fully normal spaces, *Pacific J. Math.* **2** (1952), 11–22.
3. K. Borsuk, Über Isomorphie der Funktionalräume, *Bull. Acad. Polonaise* (1933), 1–10.
4. F. Deutsch, A general existence theorem for best approximations, *in* "Approximations in Theorie und Praxis" (G. Meinardus, Ed.), pp. 71–83, Bibliographisches Institut, Mannheim, 1979.
5. F. Deutsch, Linear selections for the metric projection, *J. Functional Anal.* **99** (1982), 269–292.
6. F. Deutsch, Wu Li, and S.-H. Park, Characterizations of continuous and Lipschitz continuous metric selections in normed linear spaces, *J. Approx. Theory* **58** (1989), 297–314.
7. J. Dugundji, An extension of Tietze's theorem, *Pacific J. Math.* **1** (1951), 353–367.
8. H. Fakhoury, Projections de meilleure approximation continues dans certains espaces de Banach, *C. R. Acad. Sci. Paris* **276** (1973), 45–48.
9. R. Holmes, B. Scranton, and J. Ward, Approximation from the space of compact operators and other M-ideals, *Duke Math. J.* **42** (1975), 259–269.
10. E. Michael, Selected selection theorems, *Amer. Math. Monthly* **63** (1956), 233–237.
11. W. Rudin, "Real and Complex Analysis," McGraw–Hill, New York, 1966.
12. J. Stoer, Über die Existenz linearer Approximationsoperatoren, *in* "Funktionalanalysis, Approximationstheorie, und Numerische Mathematik" (L. Collatz, G. Meinardus, and H. Ungar, Eds.), pp. 55–57, Birkhäuser, Basel, 1967.
13. D. T. Yost, Best approximation and intersections of balls in Banach spaces, *Bull. Austral. Math. Soc.* **20** (1979), 285–300.